Three-Dimensional Figures

4

4.1

Whirlygigs for Sale!

Rotating Two-Dimensional Figures through Space

LEARNING GOALS

In this lesson, you will:

- Apply rotations to two-dimensional plane figures to create three-dimensional solids.
- Describe three-dimensional solids formed by rotations of plane figures through space.

KEY TERM

- disc

Throughout this chapter, you will analyze three-dimensional objects and solids that are "created" through transformations of two-dimensional plane figures.

But, of course, solids are not really "created" out of two-dimensional objects. How could they be? Two-dimensional objects have no thickness. If you stacked a million of them on top of each other, their combined thickness would still be zero. And translating two-dimensional figures does not really create solids. Translations simply move a geometric object from one location to another.

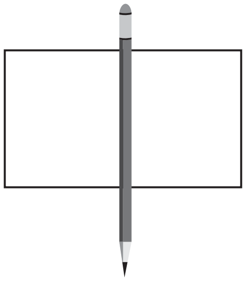
However, *thinking* about solid figures and three-dimensional objects as being "created" through transformations of two-dimensional objects is useful when you want to see how volume formulas were "created."

PROBLEM 1 Rectangular Spinners

You and a classmate are starting a summer business, making spinning toys for small children that do not require batteries and use various geometric shapes.

Previously, you learned about rotations on a coordinate plane. You can also perform rotations in three-dimensional space.

1. You and your classmate begin by exploring rectangles.
   - **a.** Draw a rectangle on an index card.
   - **b.** Cut out the rectangle and tape it along the center to a pencil below the eraser as shown.
   - **c.** Hold on to the eraser with your thumb and index finger such that the pencil is resting on its tip. Rotate the rectangle by holding on to the eraser and spinning the pencil. You can get the same effect by putting the lower portion of the pencil between both palms of your hands and rolling the pencil by moving your hands back and forth.



   - **d.** As the rectangle rotates about the pencil, the image of a three-dimensional solid is formed. Which of these solids most closely resembles the image formed by the rotating rectangle?

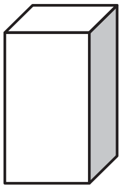


Figure 1

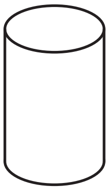


Figure 2

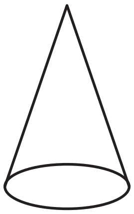


Figure 3

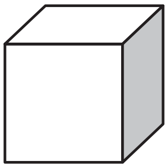


Figure 4

   - **e.** Name the solid formed by rotating the rectangle about the pencil.

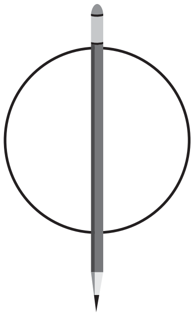
   - **f.** Relate the dimensions of the rectangle to the dimensions of this solid.

2. You and your classmate explore circles next.

   a. Draw a circle on an index card.

   b. Cut out the circle and tape it along the center to a pencil below the eraser as shown.

   c. Hold on to the eraser with your thumb and index finger such that the pencil is resting on its tip. Rotate the circle by holding on to the eraser and spinning the pencil. You can get the same effect by putting the lower portion of the pencil between both palms of your hands and rolling the pencil by moving your hands back and forth.



Remember, a circle is the set of all points that are equal distance from the center. A **disc** is the set of all points on the circle and in the interior of the circle.

   d. As the disc rotates about the pencil, the image of a three-dimensional solid is formed. Which of these solids most closely resembles the image formed by the rotating disc?

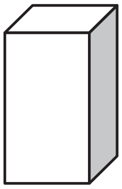


Figure 1

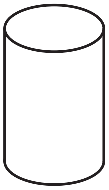


Figure 2

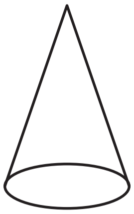


Figure 3

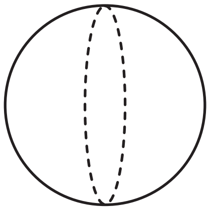


Figure 4

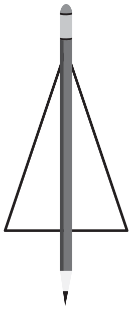
   e. Name the solid formed by rotating the circle about the pencil.

   f. Relate the dimensions of the disc to the dimensions of this solid.

3. You and your classmate finish by exploring triangles.

   a. Draw a triangle on an index card.

   b. Cut out the triangle and tape it lengthwise along the center to a pencil below the eraser as shown.



   c. Hold on to the eraser with your thumb and index finger such that the pencil is resting on its tip. Rotate the triangle by holding on to the eraser and spinning the pencil. You can get the same effect by putting the lower portion of the pencil between both palms of your hands and rolling the pencil by moving your hands back and forth.

   d. As the triangle rotates about the pencil, the image of a three-dimensional solid is formed. Which of these solids most closely resembles the image formed by the rotating triangle?

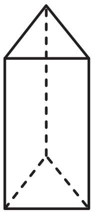


Figure 1

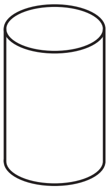


Figure 2

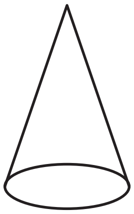


Figure 3

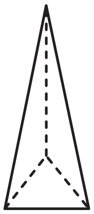


Figure 4

   e. Name the solid formed by rotating the triangle about the pencil.

   f. Relate the dimensions of the triangle to the dimensions of this solid.

Be prepared to share your solutions and methods.

4.2

Cakes and Pancakes

Translating and Stacking Two-Dimensional Figures

LEARNING GOALS

In this lesson, you will:

- Apply translations to two-dimensional plane figures to create three-dimensional solids.
- Describe three-dimensional solids formed by translations of plane figures through space.
- Build three-dimensional solids by stacking congruent or similar two-dimensional plane figures.

KEY TERMS

- isometric paper
- right triangular prism
- oblique triangular prism
- right rectangular prism
- oblique rectangular prism
- right cylinder
- oblique cylinder

You may never have heard of isometric projection before, but you have probably seen something like it many times when playing video games.

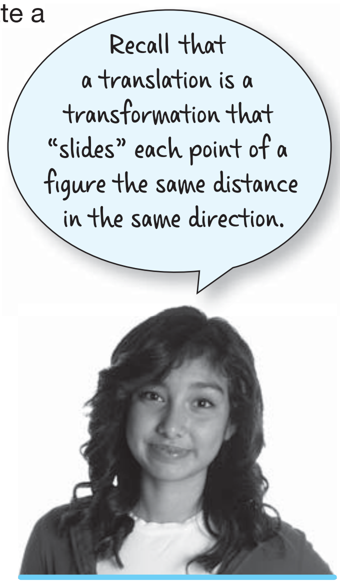
Isometric projection is used to give the environment in a video game a three-dimensional effect by rotating the visuals and by drawing items on the screen using angles of perspective.

One of the first uses of isometric graphics was in the video game Q*bert, released in 1982. The game involved an isometric pyramid of cubes. The main character, Q*bert, starts the game at the top of the pyramid and moves diagonally from cube to cube, causing them to change color. Each level is cleared when all of the cubes change color. Of course, Q*bert is chased by several enemies.

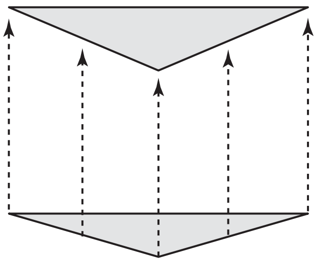
While it may seem simple now, it was extremely popular at the time. Q*bert had his own line of toys, and even his own animated television show!

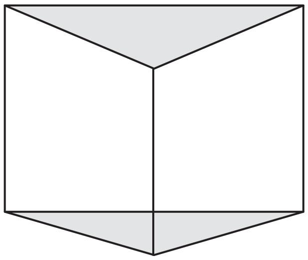
PROBLEM 1 These Figures Take the Cake

You can translate a two-dimensional figure through space to create a model of a three-dimensional figure.



1. Suppose you and your classmate want to design a cake with triangular bases. You can imagine that the bottom triangular base is translated straight up to create the top triangular base.





   a. What is the shape of each lateral face of this polyhedron formed by this translation?

   b. What is the name of the solid formed by this translation?

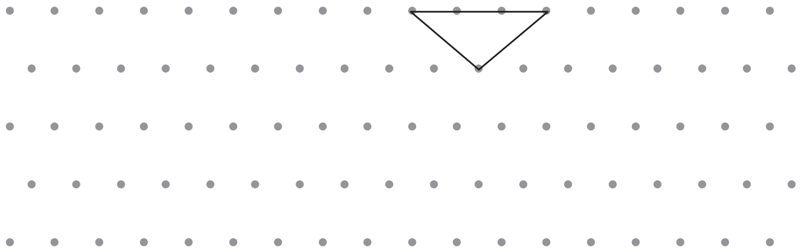
A two-dimensional representation of a triangular prism can be obtained by translating a triangle in two dimensions and connecting corresponding vertices. You can use **isometric paper**, or dot paper, to create a two-dimensional representation of a three-dimensional figure. Engineers often use isometric drawings to show three-dimensional diagrams on "two-dimensional" paper.

2. Translate each triangle to create a second triangle. Use dashed line segments to connect the corresponding vertices.

a. Translate this triangle in a diagonal direction.

b. Translate this right triangle in a diagonal direction.

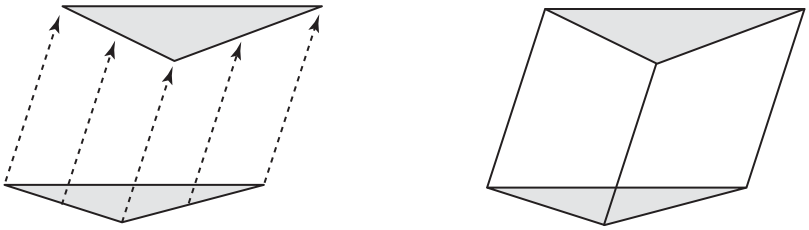
c. Translate this triangle vertically.



d. Translate this triangle horizontally.

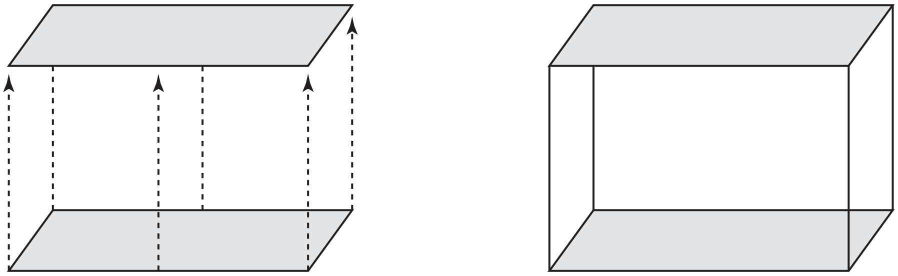
3. What do you notice about the relationship among the line segments connecting the vertices in each of your drawings?

When you translate a triangle through space in a direction that is perpendicular to the plane containing the triangle, the solid formed is a **right triangular prism**. The triangular prism cake that you and your classmate created in Question 1 is an example of a right triangular prism. When you translate a triangle through space in a direction that is *not* perpendicular to the plane containing the triangle, the solid formed is an **oblique triangular prism**. An example of an oblique triangular prism is shown.



4. What is the shape of each lateral face of an oblique triangular prism?

5. Suppose you and your classmate want to design a cake with rectangular bases. You can imagine that the bottom rectangular base is translated straight up to create the top rectangular base.



a. What is the shape of each lateral face of the solid figure formed by this translation?

b. What is the name of the solid formed by this translation?

A two-dimensional representation of a rectangular prism can be obtained by translating a rectangle in two dimensions and connecting corresponding vertices.

6. Draw a rectangle and translate it in a diagonal direction to create a second rectangle. Use dashed line segments to connect the corresponding vertices.

7. Analyze your drawing.

a. What do you notice about the relationship among the line segments connecting the vertices in the drawing?

b. What is the name of a rectangular prism that has all congruent sides?

c. What two-dimensional figure would you translate to create a rectangular prism with all congruent sides?

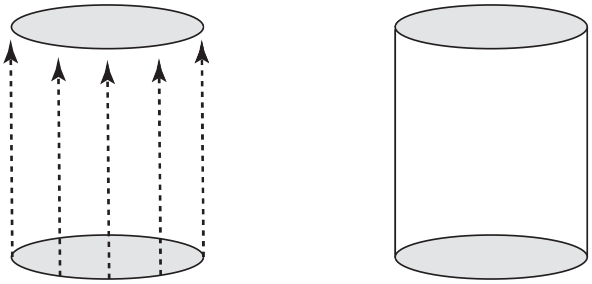
d. Sketch an example of a rectangular prism with all congruent sides.

When you translate a rectangle through space in a direction that is perpendicular to the plane containing the rectangle, the solid formed is a **right rectangular prism**. The rectangular prism cake that you and your classmate created in Question 8 is an example of a right rectangular prism. When you translate a rectangle through space in a direction that is *not* perpendicular to the plane containing the rectangle, the solid formed is an **oblique rectangular prism**.

8. What shape would each lateral face of an oblique rectangular prism be?

9. Sketch an oblique rectangular prism.

10. Suppose you and your classmate want to design a cake with circular bases. You can imagine that the bottom circular base, a disc, is translated straight up to create the top circular base.

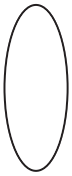


a. What shape would the lateral face of the solid figure formed by this translation be?

b. What is the name of the solid formed by this translation?

A two-dimensional representation of a cylinder can be obtained by translating an oval in two dimensions and connecting the tops and bottoms of the ovals.

11. Translate the oval in a diagonal direction to create a second oval. Use dashed line segments to connect the tops and bottoms of the ovals.



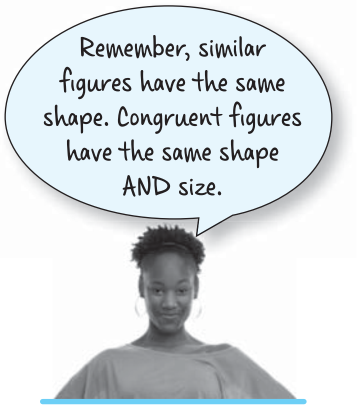
12. What do you notice about the relationship among the line segments in the drawing?

When you translate a disc through space in a direction that is perpendicular to the plane containing the disc, the solid formed is a **right cylinder**. The cylinder cake that you and your classmate created in Question 13 is an example of a right cylinder. When you translate a disc through space in a direction that is *not* perpendicular to the plane containing the disc, the solid formed is an **oblique cylinder**.

13. What shape would the lateral face of an oblique cylinder be?

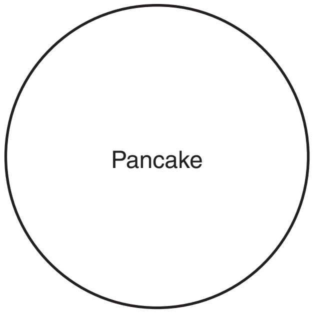
14. Sketch an oblique cylinder.

PROBLEM 2 Congruent and Similar



The math club at school is planning a pancake breakfast as a fund-raiser. Because this is a fund-raiser for the math club, the pancakes will use various geometric shapes!

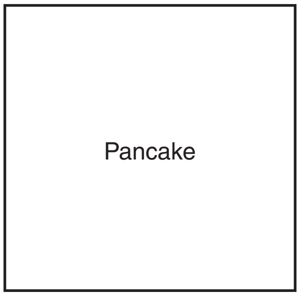
1. Imagine you stack congruent circular pancakes on top of each other.



   a. What is the name of the solid formed by this stack of pancakes?

   b. Relate the dimensions of a single circular pancake to the dimensions of this solid.

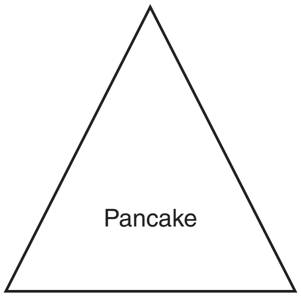
2. Imagine you stack congruent square pancakes on top of each other.



   a. What is the name of the solid formed by this stack of pancakes?

   b. Relate the dimensions of a single square to the dimensions of this solid.

3. Imagine you stack congruent triangular pancakes on top of each other.



   a. What is the name of the solid formed by this stack of pancakes?

   b. Relate the dimensions of the triangle to the dimensions of this solid.

4. What do you notice about the three-dimensional solids created by stacking congruent figures?

5. What type of solid would be formed by stacking congruent rectangles? pentagons? hexagons?

6. Imagine you stack similar circular pancakes on top of each other so that each layer of the stack is composed of a slightly smaller pancake than the previous layer.

   a. What is the name of the solid formed by this stack of pancakes?

   b. Relate the dimensions of a single pancake to the dimensions of the solid.

7. Imagine you stack similar square pancakes on top of each other so that each layer of the stack is composed of a slightly smaller pancake than the previous layer.

   a. What is the name of the solid formed by this stack of pancakes?

   b. Relate the dimensions of a single pancake to the dimensions of the solid.

8. Imagine you stack similar triangular pancakes on top of each other so that each layer of the stack is composed of a slightly smaller pancake than the previous layer.

   a. What is the name of the solid formed by this stack of pancakes?

   b. Relate the dimensions of a single pancake to the dimensions of the solid.

9. What do you notice about the three-dimensional solids created by stacking similar figures?

10. What type of solid would be formed by stacking similar rectangles? pentagons? hexagons?

11. Use what you have learned in this lesson to make an informal argument that explains the volume formulas for prisms and cylinders.

12. Complete the graphic organizer to record the volume formulas and the transformations you have used to create the solid figures.

Volume Formulas and Transformations

Prisms

$V =$ (area of base) $\times$ height

$V = Bh$

Transformations:

Pyramids

$V = \frac{1}{3}Bh$

Transformations:

Cylinders

$V = \pi r^2 h$

Transformations:

Cones

$V = \frac{1}{3}\pi r^2 h$

Transformations:

Talk the Talk

1. Which of the following actions could result in forming the same solid? Cut out the cards shown and sort them into groups that could each form the same solid figure. Then, draw an example of each solid figure and label each group. Explain how you sorted the actions. Be sure to name the solid that best represents the object.

translating an isosceles triangle	translating a right triangle	translating a square	translating a rectangle
translating a circle	rotating a rectangle	rotating a triangle	rotating a circle
stacking congruent circles	stacking similar circles	stacking congruent rectangles	stacking similar rectangles
stacking congruent squares	stacking similar squares	stacking congruent triangles	stacking similar triangles

4

Be prepared to share your solutions and methods.

4

4.3

Cavalieri's Principles

Application of Cavalieri's Principles

LEARNING GOALS

In this lesson, you will:

- Explore Cavalieri's principle for two-dimensional geometric figures (area).
- Explore Cavalieri's principle for three-dimensional objects (volume).

KEY TERM

- Cavalieri's principle

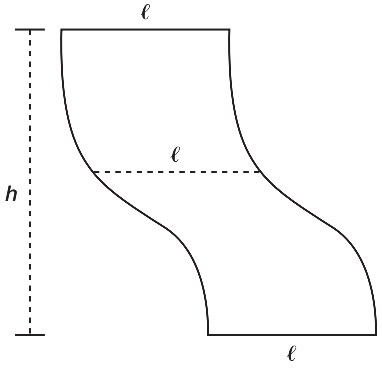
Bonaventura Cavalieri was an Italian mathematician who lived from 1598 to 1647. Cavalieri is well known for his work in geometry as well as optics and motion.

His first book dealt with the theory of mirrors shaped into parabolas, hyperbolas, and ellipses. What is most amazing about this work is that the technology to create the mirrors that he was writing about didn't even exist yet!

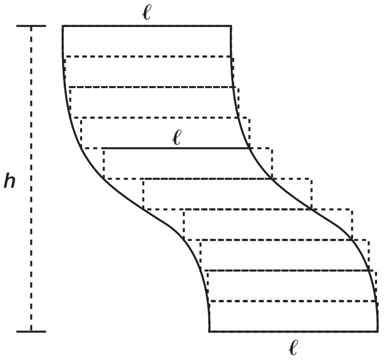
Cavalieri is perhaps best known for his work with areas and volumes. He is so well known that he even has a principle named after him—Cavalieri's principle.

PROBLEM 1 Approximating the Area of a Two-Dimensional Figure

One strategy for approximating the area of an irregularly shaped figure is to divide the figure into familiar shapes and determine the total area of all of the shapes. Consider the irregular shape shown. The distance across any part of the figure is the same.



1. You can approximate the area by dividing the irregular shape into congruent rectangles. To start, let's divide this shape into 10 congruent rectangles.



   a. What is the length, the height, and the area of each congruent rectangle?

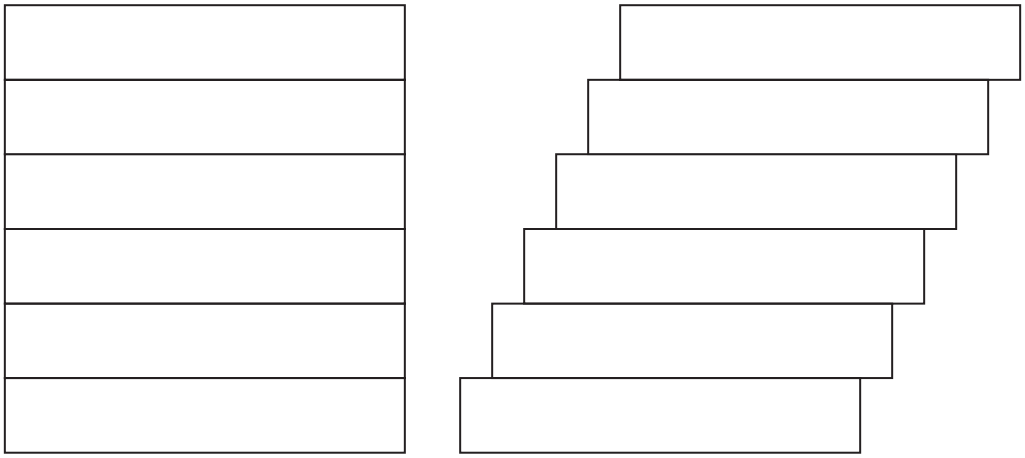
   b. What is the approximate area of the irregularly shaped figure?

2. If this irregularly shaped figure were divided into 1000 congruent rectangles, what would be the area of each congruent rectangle? What would be the approximate area of the figure?

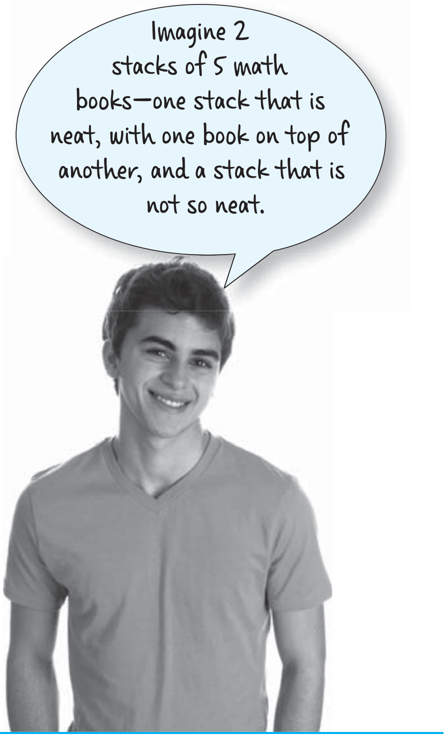
3. If this irregularly shaped figure were divided into n congruent rectangles, what would be the area of each congruent rectangle? What would be the approximate area of the figure?

4. If the irregularly shaped figure were divided into only one rectangle, what would be the approximate area of the figure?

5. Compare the area of the two figures shown. Each rectangle has a height of h and a base equal to length ℓ.

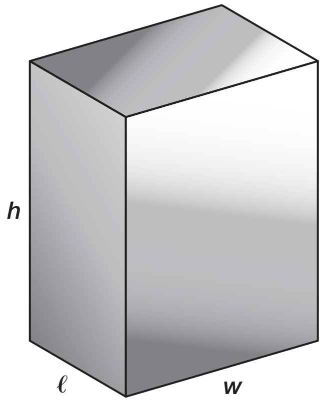


You have just explored **Cavalieri's principle** for two-dimensional figures, sometimes called the method of indivisibles. If the lengths of one-dimensional slices—just a line segment—of the two figures are the same, then the figures have the same area. This is best illustrated by making several slices to one figure and pushing them to the side to form a second figure.

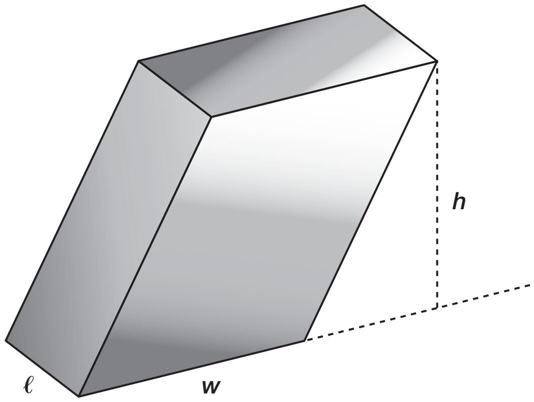


PROBLEM 2 Cavalieri's Principle for Volume

Consider the right rectangular prism and the oblique rectangular prism shown.



Right rectangular prism



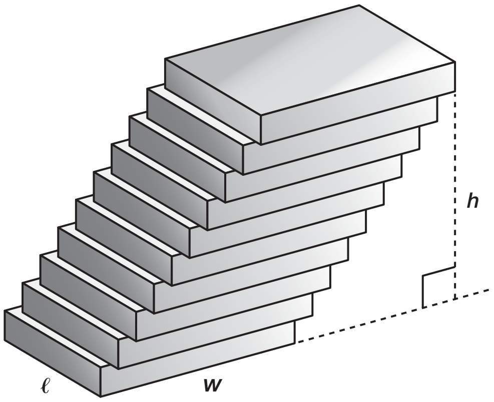
Oblique rectangular prism

1. What geometric figure represents a cross section of each that is perpendicular to the base?

2. What are the dimensions of one cross section?

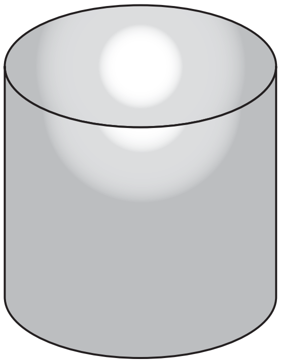
3. What is the volume of the right rectangular prism?

4. Approximate the volume of the oblique rectangular prism by dividing the prism into ten congruent right prisms as shown.

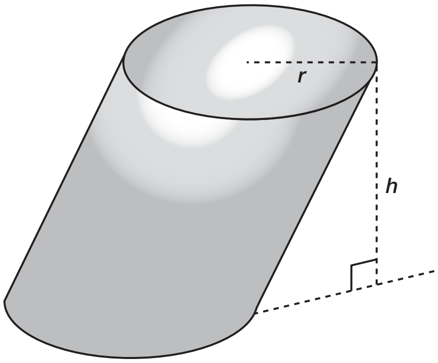


You have just explored Cavalieri's principle for three-dimensional figures. Given two solids included between parallel planes, if every plane cross section parallel to the given planes has the same area in both solids, then the volumes of the solids are equal. In other words, if, in two solids of equal altitude, the sections made by planes parallel to and at the same distance from their respective bases are always equal, then the volumes of the two solids are equal.

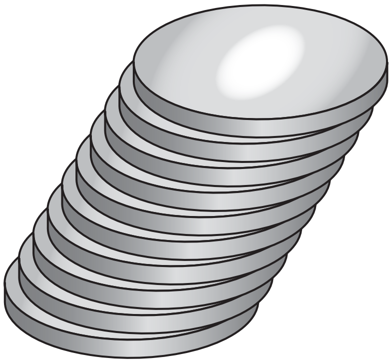
For a second example of this principle, consider a right cylinder and an oblique cylinder having the same height and radii of equal length.



Right cylinder



Oblique cylinder



5. What geometric figure best represents the ten cross sections of the cylinders?

6. What are the dimensions of one cross section?

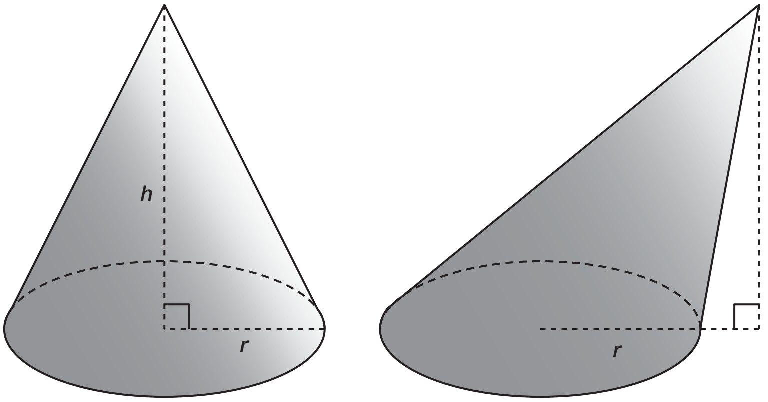
7. What is the volume of one cross section?

8. What is the volume of the oblique cylinder?

9. What is the volume of the right cylinder?

You have just shown the volume of a right cylinder and the volume of an oblique cylinder are equal, provided both cylinders have the same height and radii of equal length.

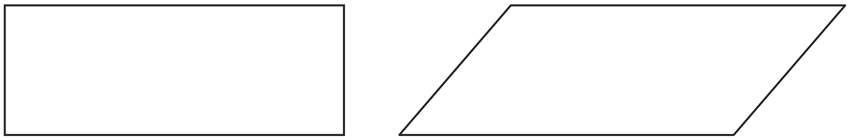
10. Using Cavalieri's principle, what can you conclude about the volume of these two cones, assuming the heights are equal and the radii in each cone are congruent?



It is important to mention that the Cavalieri principles do not compute exact volumes or areas. These principles only show that volumes or areas are equal without computing the actual values. Cavalieri used this method to relate the area or volume of one unknown object to one or more objects for which the area or volume could be determined.

4 Talk the Talk

1. Consider the rectangle and the parallelogram shown to be of equal height with bases of the same length.



Knowing the area formula for a rectangle, how is Cavalieri's principle used to determine the area formula for the parallelogram?

Be prepared to share your solutions and methods.

4.4

Spin to Win

Volume of Cones and Pyramids

LEARNING GOALS

In this lesson, you will:

- Rotate two-dimensional plane figures to generate three-dimensional figures.
- Give an informal argument for the volume of cones and pyramids.

Imagine that you are, right now, facing a clock and reading the time on that clock—let's imagine that it's 2:28.

Now imagine that you are blasted away from that clock at the speed of light, yet you are still able to read the time on it (of course you wouldn't be able to, really, but this is imagination!).

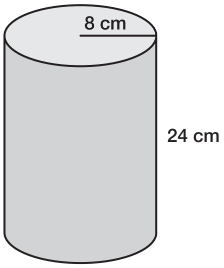
What time would you see on the clock as you traveled away from it at the speed of light? The light bouncing off the clock travels at the speed of light, so, as you travel farther and farther away from the clock, all you could possibly see was the time on the clock as it was right when you left.

Einstein considered this "thought experiment" among many others to help him arrive at groundbreaking theories in physics.

PROBLEM 1 Building Cylinders

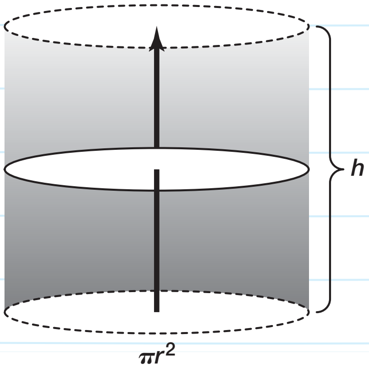
You have learned that when two-dimensional shapes are translated, rotated, or stacked, they can form three-dimensional solids. You can also use transformations and stacking to build formulas for three-dimensional figures.

1. Determine the volume for the cylinder shown. Show your work.



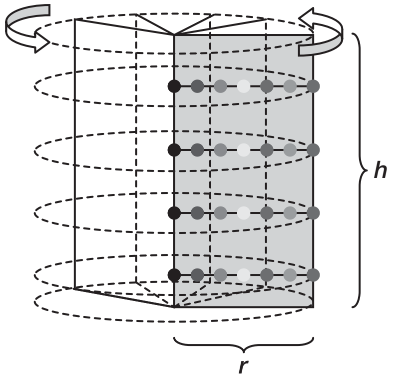
To derive the formula for the volume of a cylinder, you can think of the cylinder as an infinite stack of discs, each with an area of πr^2. These discs are stacked to a height of *h*, the height of the cylinder.

So, the volume of the cylinder is $\pi r^2 \times h$, or $\pi r^2 h$.

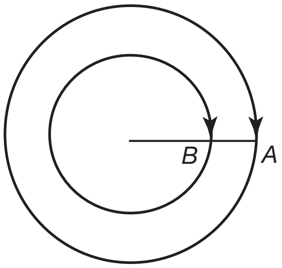


2. Can you choose any disc in the cylinder and multiply the area by the height to calculate the volume of the cylinder? Explain your reasoning.

Another way to think about a cylinder is the rotation of a rectangle about one of its sides as shown. The rotation of the set of points that make up the rectangle forms the cylinder.

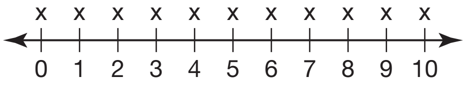


To determine the volume of the cylinder, you can multiply the area of the rectangle by the distance that the points of the rectangle rotate. However, the points of the rectangle don't all rotate the same distance. Consider a top view of the cylinder. The distance that point *A* rotates is greater than the distance that point *B* rotates.



You can't calculate the distance that each point rotates because there are an infinite number of points, each rotating a different distance. But you can use an average, or typical point, of the rectangle.

3. Consider the dot plot shown.



a. What is the median of the data?

b. Describe how you could determine the median of these data without doing any calculation.

4. What is the location of the average, or typical, point of the rectangle in terms of the radius and the height? Explain your reasoning.

5. What is the area of the rectangle that is rotated?

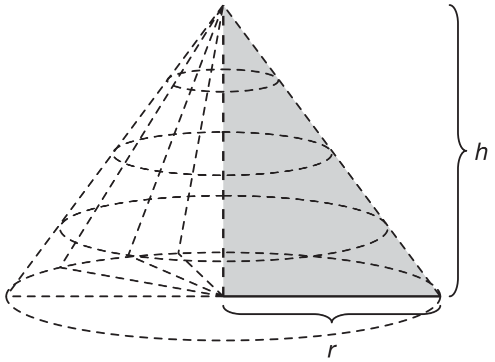
6. Use the average point of the rectangle to calculate the average distance that the points of the rectangle rotate. Explain your reasoning.

7. To determine the volume of the cylinder, multiply the area of the rectangle by the average distance that the points of the rectangle rotate. Calculate the volume of the cylinder.

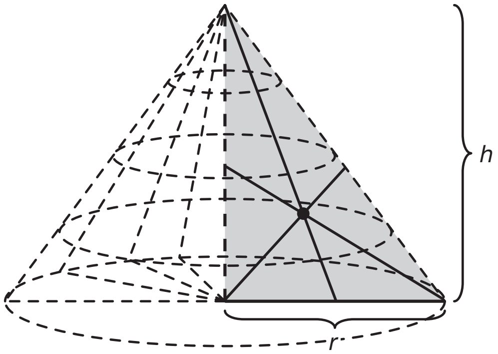
8. Compare the volume that you calculated in Question 2 to the volume that you calculated in Question 7. What do you notice?

PROBLEM 2 Building Cones

You can also think about a cone as a rotation of a right triangle about one of its legs. You can use the same strategy of determining the average, or typical, point in a right triangle to derive the formula for the volume of a cone. The figure shown represents a cone formed by rotating a right triangle about one of its legs.

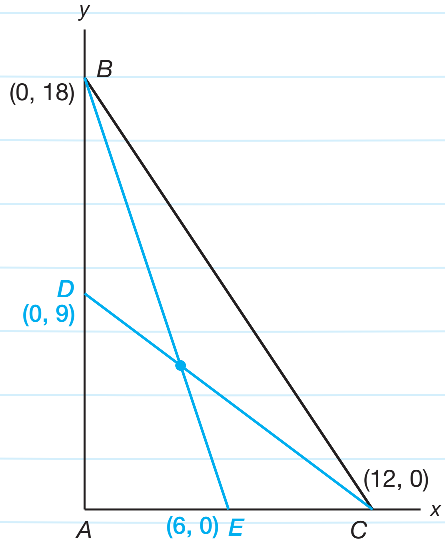


Previously, you explored points of concurrency of a triangle. Recall that the centroid is the point of concurrency of the three medians of a triangle. A median connects a vertex to the midpoint of the opposite side. You can use the centroid as the typical point of the triangle to derive the volume. The centroid of the right triangle is shown.



Let's review how to calculate the coordinates of the centroid of triangle *ABC*.

Triangle *ABC* has vertices at *A* (0, 0), *B* (0, 18), and *C* (12, 0).



First, determine the locations of the midpoints of side *AB* and side *AC*.

- Midpoint of side *AC*: $\left(\frac{0+12}{2}, \frac{0+0}{2}\right)$, or (6, 0)
- Midpoint of side *AB*: $\left(\frac{0+0}{2}, \frac{0+18}{2}\right)$, or (0, 9)

Why do we need to draw only 2 median lines instead of all 3?

Next, determine an equation representing each median.

- The slope of $\overline{BE}$ is -3, and the *y*-intercept is 18.

  Slope of $\overline{BE} = \frac{0-18}{6-0} = -\frac{18}{6} = -3$

  So, the equation of $\overline{BE}$ is $y = -3x + 18$.

- The slope of $\overline{CD}$ is $-\frac{3}{4}$, and the *y*-intercept is 9.

  Slope of $\overline{CD} = \frac{0-9}{12-0} = -\frac{9}{12} = -\frac{3}{4}$

  So, the equation of $\overline{CD}$ is $y = -\frac{3}{4}x + 9$.

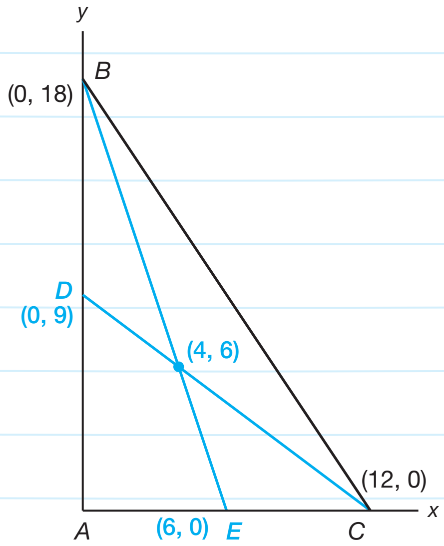
Solve the system of equations to determine the coordinates of the centroid.

$$\begin{cases} y = -3x + 18 \\ y = -\frac{3}{4}x + 9 \end{cases}$$

$$-\frac{3}{4}x + 9 = -3x + 18$$
$$-\frac{3}{4}x = -3x + 9$$
$$\frac{1}{4}x = x - 3$$
$$x = 4$$

$$y = -3x + 18$$
$$y = -3(4) + 18$$
$$y = 6$$

The centroid of the triangle is located at (4, 6).



1. What is the area of triangle *ABC*?

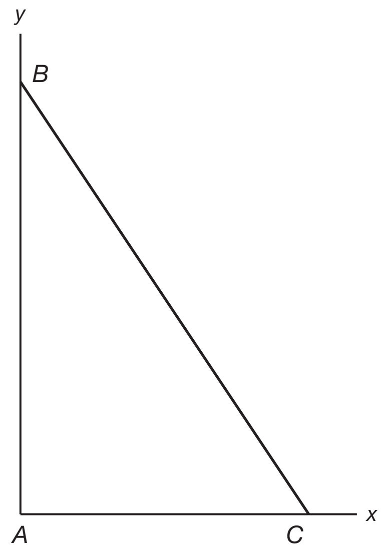
2. What is the average distance that all the points of the triangle are rotated? Show your work and explain your reasoning.

3. Determine the volume of the cone. Show your work and explain your reasoning.

4. Use the formula for the volume of a cone, $\frac{1}{3}\pi r^2 h$, to calculate the volume of the cone. Show your work.

5. Compare the volume that you calculated by rotating the right triangle to the volume that you calculated using the formula for the volume of a cone. What do you notice?

6. Derive the formula for the volume of any cone with radius, r, and height, h, by rotating a right triangle with vertices at $(0, 0)$, $(0, h)$, and $(r, 0)$.



PROBLEM 3 And Now, Pyramids

You can use what you know about the similarities and differences between cylinders and cones to make conjectures about the volume of pyramids.

1. Compare different ways to create cylinders and cones. What similarities and differences are there between creating cylinders and cones:

   a. by stacking?

   b. by rotating?

2. Analyze the formulas for the volumes of the cylinder and cone.

$$\text{Volume of cylinder} = \pi r^2 h$$

$$\text{Volume of cone} = \frac{1}{3}\pi r^2 h$$

   a. Which part of each formula describes the area of the base of each solid figure, *B*? Explain why.

   b. Which part of each formula describes the height of each solid figure?

   c. Rewrite the volume formulas for each solid figure using the variables *B* and *h*.

3. How are the formulas for the volumes of a cylinder and cone similar and different?

4. Now compare different ways to create prisms and pyramids. What similarities and differences are there between creating prisms and pyramids:

a. by stacking?

b. by rotating?

5. Analyze the formula for the volume of a prism.

Volume of prism: $V = Bh$

a. Which part of the formula represents the area of the base?

b. Which part of the formula represents the height?

6. Based on your answers to Questions 1 through 5, what conjecture can you make about the formula for the volume of any pyramid? Explain your reasoning.

Be prepared to share your solutions and methods.

4

4.5

Spheres à la Archimedes

Volume of a Sphere

LEARNING GOAL

In this lesson, you will:

- Derive the formula for the volume of a sphere.

KEY TERMS

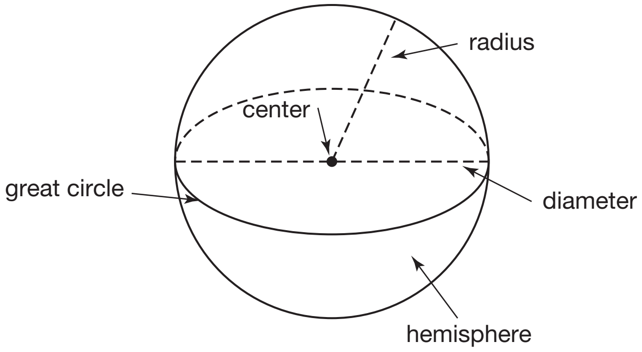
- sphere
- radius of a sphere
- diameter of a sphere
- great circle of a sphere
- hemisphere
- annulus

Archimedes of Syracuse, Sicily, who lived from 287 BC to 212 BC, was an ancient Greek mathematician, physicist, and engineer. Archimedes discovered formulas for computing volumes of spheres, cylinders, and cones.

Archimedes has been honored in many ways for his contributions. He has appeared on postage stamps in East Germany, Greece, Italy, Nicaragua, San Marino, and Spain. His portrait appears on the Fields Medal for outstanding achievement in mathematics. You can even say that his honors are out of this world. There is a crater on the moon named Archimedes, a mountain range on the moon named the Montes Archimedes, and an asteroid named 3600 Archimedes!

PROBLEM 1 Starting with Circles . . . and Cones

Recall that a circle is the set of all points in two dimensions that are equidistant from the center of the circle. A sphere can be thought of as a three-dimensional circle.



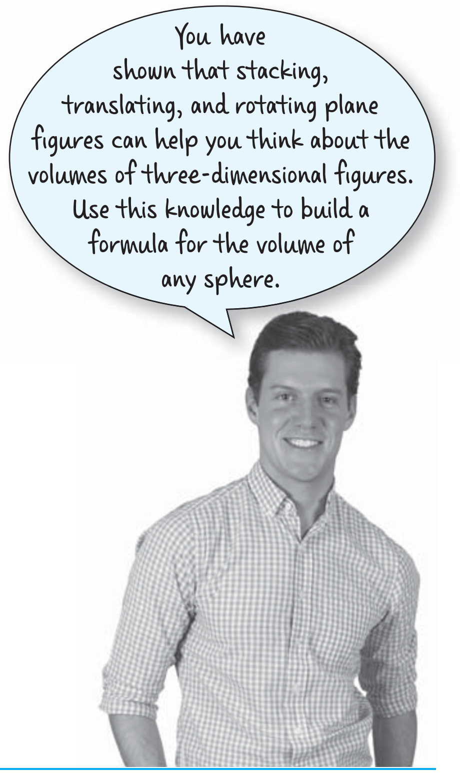
A **sphere** is the set of all points in three dimensions that are equidistant from a given point called the center.

The **radius of a sphere** is a line segment drawn from the center of the sphere to a point on the sphere.

The **diameter of a sphere** is a line segment drawn between two points on the sphere passing through the center.

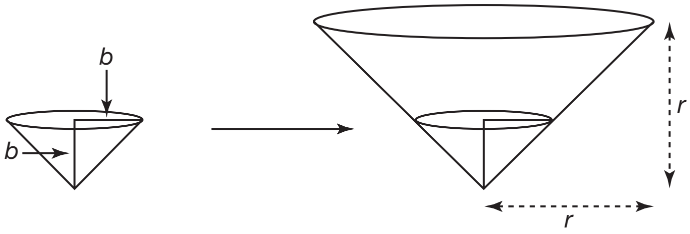
A **great circle of a sphere** is a cross section of a sphere when a plane passes through the center of the sphere.

A **hemisphere** is half of a sphere bounded by a great circle.



The cone shown on the left has a height and a radius equal to b. The height and the radius form two legs of a right triangle inside the cone. The hypotenuse lies along the side of the cone.

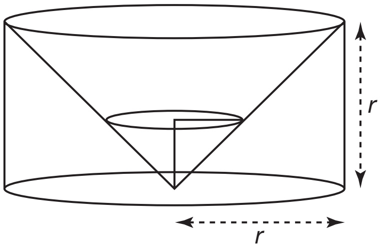
The cone shown on the right is an enlargement of the first cone. It also has a height that is equal to its radius, r. The smaller cone is shown inside the larger cone.



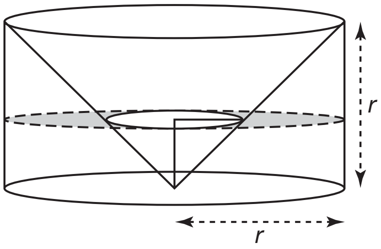
1. Write an expression to describe the area of:
   a. the base of the smaller cone.

   b. the base of the larger cone.

2. Place both cones inside a cylinder with the same radius and height as the larger cone.

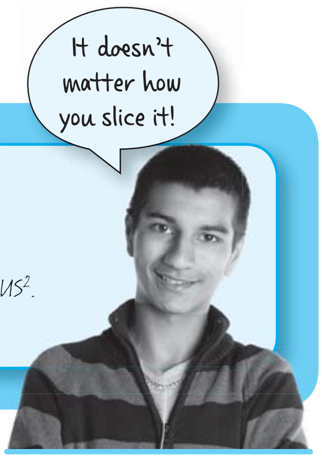


Then, make a horizontal cross section through the cylinder just at the base of the smaller cone.



What is the area of this cross section? Explain your reasoning.

3. Amy makes the following statement about the horizontal cross section of a cylinder.

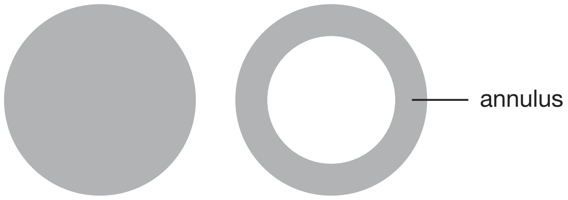


Amy

It doesn't matter where you make the cross section in a cylinder. The cross section will always be a circle with an area of $\pi \times \text{radius}^2$.

Explain why Amy is correct.

The image on the left shows the cross section of the cylinder, including the base of the cone. The image on the right shows the cross section of the cylinder with the base of the cone removed. The area bound between the two concentric circles shown on the right is called the **annulus**.



4. Calculate the area of the annulus of the cylinder. Explain your reasoning.

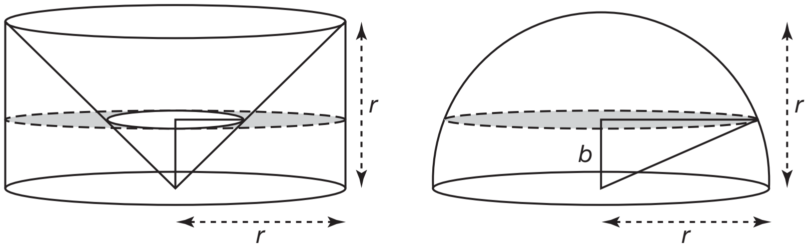
5. Show how you can use the Distributive Property to rewrite your expression from Question 4.

Now that you have a formula that describes the area of the annulus of the cylinder, let's compare this with a formula that describes the area of a cross section of a hemisphere with the same height and radius as the cylinder.

PROBLEM 2 And Now Hemispheres

The diagram shows a hemisphere with the same radius and height of the cylinder from Problem 1.

The diagram also shows that there is a cross section in the hemisphere at the same height, *b*, as that in the cylinder.



1. Describe the shape of the cross section shown in the hemisphere.

2. Analyze the hemisphere. Write expressions for the side lengths of the right triangle in the diagram. Label the diagram with the measurements.

3. Lacy says that the length of the horizontal side has a measure of *r*. Is Lacy correct? Explain your reasoning.

4. Write an expression to describe the area of the cross section in the hemisphere. Explain your reasoning.

5. Compare the area of the cross section of the hemisphere to the area of the annulus of the cylinder. What do you notice?

6. Stacy says that the volume of the cylinder and the volume of the hemisphere are not the same. But, if you remove the volume of the cone from the volume of the cylinder, then the resulting volume would be the same as the volume of the hemisphere. Is Stacy correct? Explain why or why not.

7. Write the formula for the volume of a sphere. Show your work and explain your reasoning.

Be prepared to share your solutions and methods.

Turn Up the . . .

Using Volume Formulas

4.6

LEARNING GOAL

In this lesson, you will:

- Apply the volume formulas for a pyramid, a cylinder, a cone, and a sphere to solve problems.

A mnemonic is a device used to help you remember something. For example, the mnemomic “My Very Energetic Mother Just Served Us Noodles” can be used to remember the order of the planets in the Solar System.

What about volume formulas? Can you come up with mnemonics to remember these so you don’t have to look them up?

Maybe you can use this one to remember the formula for the volume of a cylinder:

Cylinders are: “Perfectly Ready 2 Hold”

$$\begin{array}{ccccc} \text{Perfectly} & & \text{Ready 2} & & \text{Hold} \\ \downarrow & & \downarrow & & \downarrow \\ \pi & \cdot & r^2 & \cdot & h \end{array}$$

Try to come up with mnemonics for the other volume formulas that you have learned!

PROBLEM 1 On a Roll

1. A standard sized sheet of paper measures 8.5 inches by 11 inches. Use two standard sized sheets of paper to create two cylinders. One cylinder should have a height of 11 inches and the other cylinder should have a height of 8.5 inches.

2. Carol predicts that the cylinder with a height of 11 inches has a greater volume.
Lois predicts that the cylinder with a height of 8.5 inches has a greater volume.
Stu predicts that the two cylinders have the same volume.

   Predict which cylinder has the greatest volume.

3. Determine the radius and the height of each cylinder without using a measuring tool.

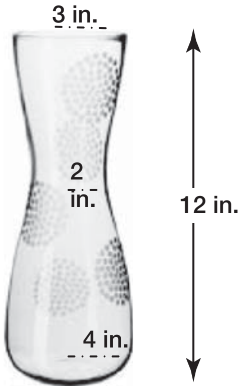
4. Calculate the volume of each cylinder to prove or disprove your prediction and determine who was correct.

5. Does the radius or the height have a greater impact on the magnitude of the volume? Explain your reasoning.

6. Consider the volume of the cylinder with a height of 8.5 inches. What radius would be required to create a cylinder with a height of 11 inches that has the same volume?

PROBLEM 2 Let's Vase It

1. Describe a strategy for approximating the volume of the vase shown.



2. Determine the approximate volume of the vase.

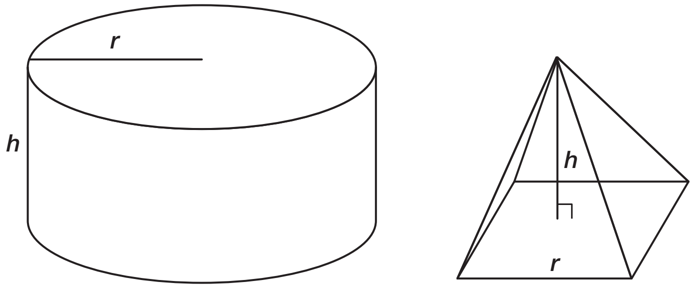
PROBLEM 3 Balloons, Lakes, and Graphs

1. A typical hot-air balloon is about 75 feet tall and about 55 feet in diameter at its widest point. About how many cubic feet of hot air does a typical hot-air balloon hold? Explain how you determined your answer.

2. Lake Erie, the smallest of the Great Lakes by volume, still holds an impressive 116 cubic miles of water. Suppose you start today dumping out the entire volume of Lake Erie using a cone cup. A typical cone cup has a diameter of $2\frac{3}{4}$ inches and a height of 4 inches. About how long would it take you to empty the lake if you could dump out one cup per second?

3. A cone and a sphere each have a radius of r units. The cone and the sphere also have equal volumes. Describe the height of the cone in terms of the radius. Show your work.

4. The diagram shows a cylinder and a square pyramid with the same height. The width of the pyramid is equal to the radius of the cylinder. Suppose that the radius of the cylinder is gradually increased and the side lengths of the square pyramid are also gradually increased. Which solid's volume would increase more rapidly? Explain your reasoning.



Be prepared to share your solutions and methods.

4.7

Tree Rings

Cross Sections

LEARNING GOALS

In this lesson, you will:

- Determine the shapes of cross sections.
- Determine the shapes of intersections of solids and planes.

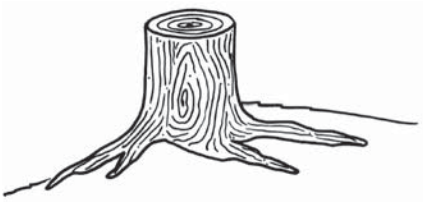
Each year, a tree grows in diameter. The amount of growth of the diameter depends on the weather conditions and the amount of water that is available to the tree. If a tree is cut perpendicular to its trunk, you can see rings, one for each year of growth. The wider the ring, the greater the amount of growth in one year.

Dendrochronology, or tree-ring dating, is the method of dating trees based on an analysis of their rings. Often times, it is possible to date a tree to an exact year.

The oldest known tree in the world is a Great Basin bristlecone pine located in White Mountains, California. It is estimated to be over 5000 years old!

PROBLEM 1 Cutting a Tree Trunk

A section of a tree trunk is roughly in the shape of a cylinder as shown.



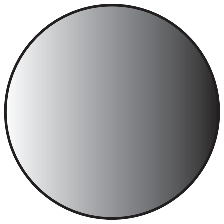
When a tree trunk is cut in order to see the tree rings, a cross section of the trunk is being studied.

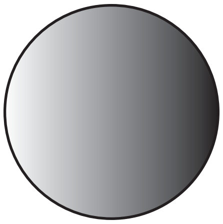
1. Suppose a plane intersects a cylinder parallel to its bases. What is the shape of the cross section? Sketch an example of this cross section.

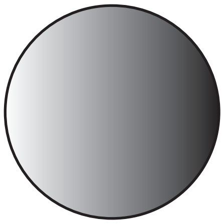
2. Suppose a plane intersects a cylinder perpendicular to its bases so that the plane passes through the centers of the bases. What is the shape of this cross section? Sketch an example of this cross section.

3. Suppose a plane intersects a cylinder so that it is not parallel to its bases. What is the shape of this cross section? Sketch an example of this cross section.

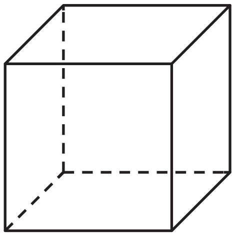
4. Consider a sphere. Sketch and describe three different cross sections formed when a plane intersects a sphere.

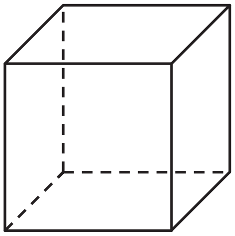


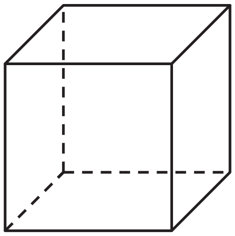


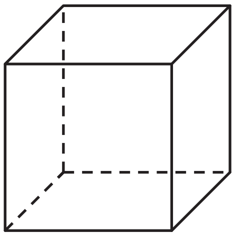


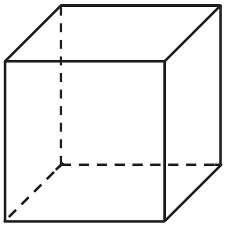
5. Consider a cube. Sketch and describe five different cross sections formed when a plane intersects a cube.





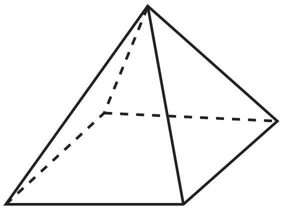


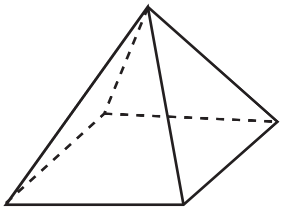


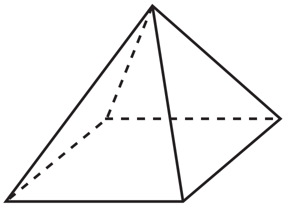


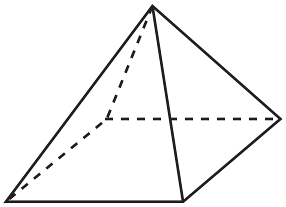
Sally says that it's not possible to form a circle or an octagon as a cross section of a cube. Is Sally correct? Explain your reasoning.

6. Consider a square pyramid. Sketch and describe four different cross sections formed when a plane intersects a square pyramid.

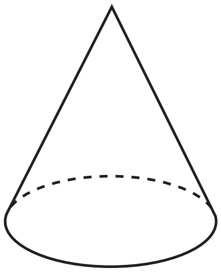


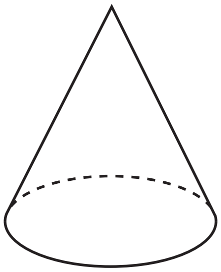


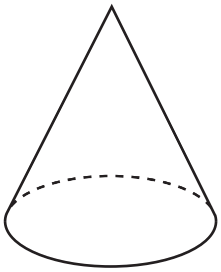


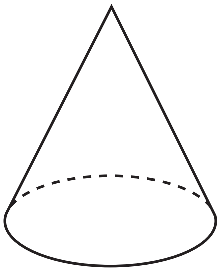


7. Consider a cone. Sketch and describe four different cross sections formed when a plane intersects a cone.



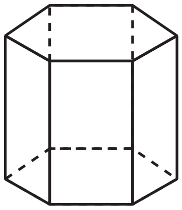




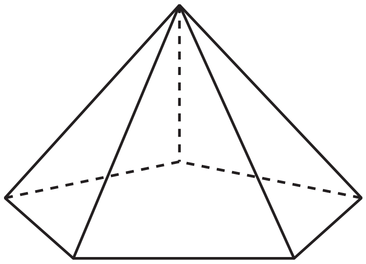


8. Use each solid to create two cross sections. Create one cross section parallel to a base and a second cross section perpendicular to a base. Then, identify each of the cross sections.

a.

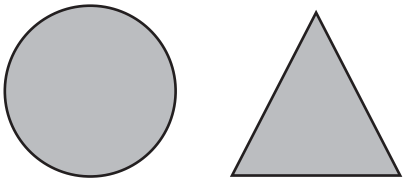


b.

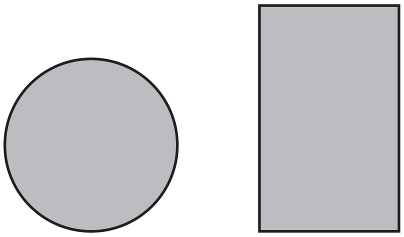


9. Draw a solid that could have the given cross sections.

   a. A cross section parallel to a base and a cross section perpendicular to a base



   b. A cross section parallel to a base and a cross section perpendicular to a base



   c. Consider a prism placed on a table such that the base is horizontal. If a plane passes through the prism horizontally at any height, describe the cross sections.

   d. Define a cylinder in terms of its cross sections.

Be prepared to share your solutions and methods.

Two Dimensions Meet Three Dimensions

4.8

Diagonals in Three Dimensions

LEARNING GOALS

In this lesson, you will:

- Use the Pythagorean Theorem to determine the length of a diagonal of a solid.
- Use a formula to determine the length of a diagonal of a rectangular solid given the lengths of three perpendicular edges.
- Use a formula to determine the length of a diagonal of a rectangular solid given the diagonal measurements of three perpendicular sides.

There are entire industries dedicated to helping people move from one location to another. Some people prefer to hire a moving company that will pack, move, and unpack all of their belongings. Other people choose to rent a van and do all the packing and unpacking themselves.

One of the most common questions heard during a move is, “Will it fit?” Moving companies pride themselves on being able to pack items as efficiently as possible. Sometimes it almost seems impossible to fit so much into so little a space.

What strategies would you use if you had to move and pack yourself?

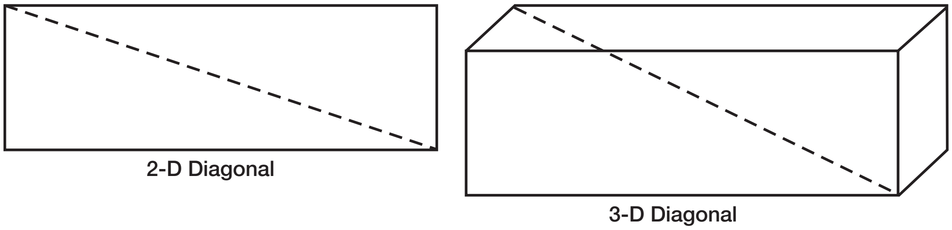
PROBLEM 1 A Box of Roses

The dimensions of a rectangular box for long-stem roses are 18 inches in length, 6 inches in width, and 4 inches in height.

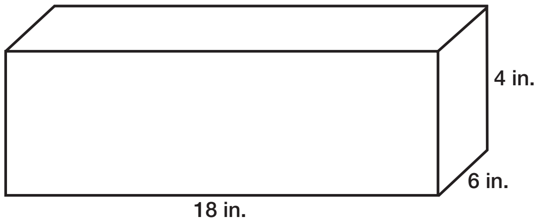
You need to determine the maximum length of a long-stem rose that will fit in the box without bended the rose's stem. You can use the Pythagorean Theorem to solve this problem.

1. What makes this problem different from all of the previous applications of the Pythagorean Theorem?

2. Compare a two-dimensional diagonal to a three-dimensional diagonal. Describe the similarities and the differences.



3. Draw all of the sides in the rectangular solid you cannot see using dotted lines.



4. Draw a three-dimensional diagonal in the rectangular solid shown.

5. If the three-dimensional diagonal is the hypotenuse and an edge of the rectangular solid is a leg of the right triangle, where is the second leg?

6. Draw the second leg using a dotted line.

7. Determine the length of the second leg.

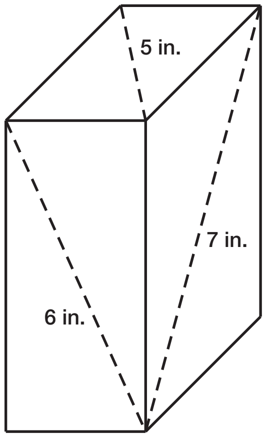
8. Determine the length of the three-dimensional diagonal.

9. Describe how you used the Pythagorean Theorem to calculate the length of the three-dimensional diagonal.

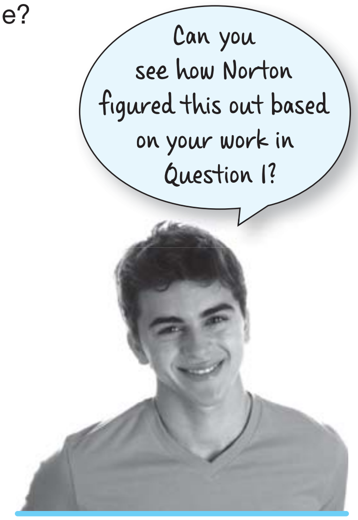
10. Norton tells his teacher he knows a shortcut for determining the length of a three-dimensional diagonal. He says, “All you have to do is calculate the sum of the squares of the rectangular solids’ three perpendicular edges [the length, the width, and the height] and that sum would be equivalent to the square of the three-dimensional diagonal.” Does this work? Use the rectangular solid in Question 3 and your answer in Question 8 to determine if Norton is correct. Explain your reasoning.

PROBLEM 2 More Solids

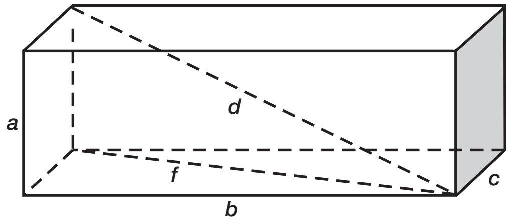
1. A rectangular prism is shown. The diagonal across the front panel is 6 inches, the diagonal across the side panel is 7 inches, and a diagonal across the top panel is 5 inches. Determine the length of a three-dimensional diagonal.



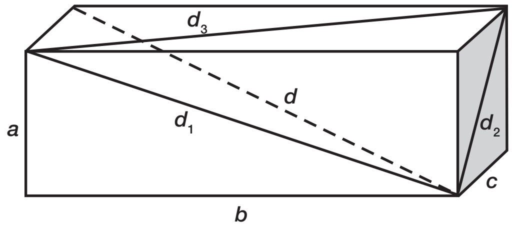
2. Norton tells his teacher that there is a much faster way to do Question 1. This time, he says, all you have to do is take one-half the sum of the squares of the diagonals on each dimension of the rectangular solid, and that would be equivalent to the square of the three-dimensional diagonal. Is Norton correct this time?



3. Write a formula to determine the three-dimensional diagonal in the rectangular prism in terms of the dimensions of the rectangular prism.



4. Write a formula to determine the three-dimensional diagonal in the rectangular prism in terms of the lengths of the diagonals.

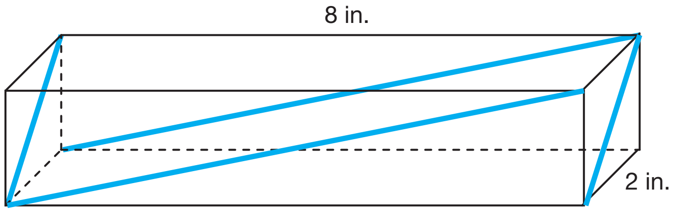


PROBLEM 3 Box It Up

1. Your new part-time job with a landscaping business requires you to transport small trees in your own car. One particular tree measures 8 feet from the root ball to the top. The interior of your car is 62 inches in length, 40 inches in width, and 45 inches in height. Determine if the tree will fit inside your car. Explain your reasoning.

2. Andy's company is bidding on a project to create a decal design for pencil boxes. If the client likes Andy's design then they will order 200 boxes. The design plan with the decal stripes is shown. The length of the three-dimensional diagonal is approximately 8.5 inches.

If the striping will cost Andy $0.59 per linear foot, how much should Andy bid to make sure he gets at least a 20% profit?



Be prepared to share your solutions and methods.

Chapter 4 Summary

KEY TERMS

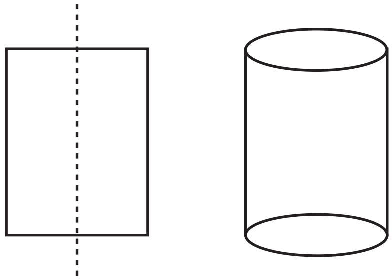
- disc (4.1)
- isometric paper (4.2)
- right triangular prism (4.2)
- oblique triangular prism (4.2)
- right rectangular prism (4.2)
- oblique rectangular prism (4.2)
- right cylinder (4.2)
- oblique cylinder (4.2)
- Cavalieri's principle (4.3)
- sphere (4.5)
- radius of a sphere (4.5)
- diameter of a sphere (4.5)
- great circle of a sphere (4.5)
- hemisphere (4.5)
- annulus (4.5)

4.1 Creating and Describing Three-Dimensional Solids Formed by Rotations of Plane Figures through Space

A three-dimensional solid is formed by rotating a two-dimensional figure around an axis. As the shape travels around in a somewhat circular motion through space, the image of a three-dimensional solid is formed. A rotated rectangle or square forms a cylinder. A rotated triangle forms a cone. A rotated disc forms a sphere. The radius of the resulting solid relates to the distance of the edge of the plane figure to the axis point.

Example

A cylinder is formed by rotating a rectangle around the axis. The width of the rectangle is equal to the radius of the cylinder's base.

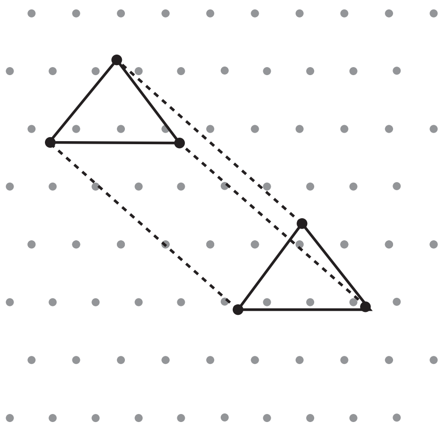


4.2 Creating and Describing Three-Dimensional Solids Formed by Translations of Plane Figures through Space

A two-dimensional drawing of a solid can be obtained by translating a plane figure through two dimensions and connecting the corresponding vertices. The solid formed by translating a rectangle is a rectangular prism. The solid formed by translating a square is a cube. The solid formed by translating a triangle is a triangular prism. The solid formed by translating a circle is a cylinder.

Example

A triangular prism is formed by translating a triangle through space.

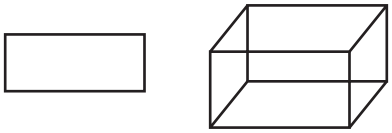


4.2 Building Three-Dimensional Objects by Stacking Congruent Plane Figures

Congruent shapes or figures are the same shape and the same size. A stack of congruent figures can form a solid shape. A stack of congruent circles forms a cylinder. A stack of congruent squares forms a rectangular prism or cube. A stack of congruent triangles forms a triangular prism. The dimensions of the base of the prism or cylinder are the same as the original plane figure.

Example

A rectangular prism can be formed by stacking rectangles.

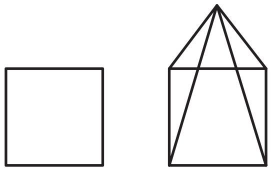


4.2 Building Three-Dimensional Objects by Stacking Similar Plane Figures

Similar shapes or figures have the same shape, but they do not have to be the same size. A stack of similar figures that is incrementally smaller with each layer form a solid shape. A stack of similar circles forms a cone. A stack of similar squares, rectangles, triangles, pentagons, hexagons, or other polygons forms a pyramid. The dimensions of the base of the cone or pyramid are the same as the original plane figure.

Example

A square pyramid is formed by stacking similar squares.



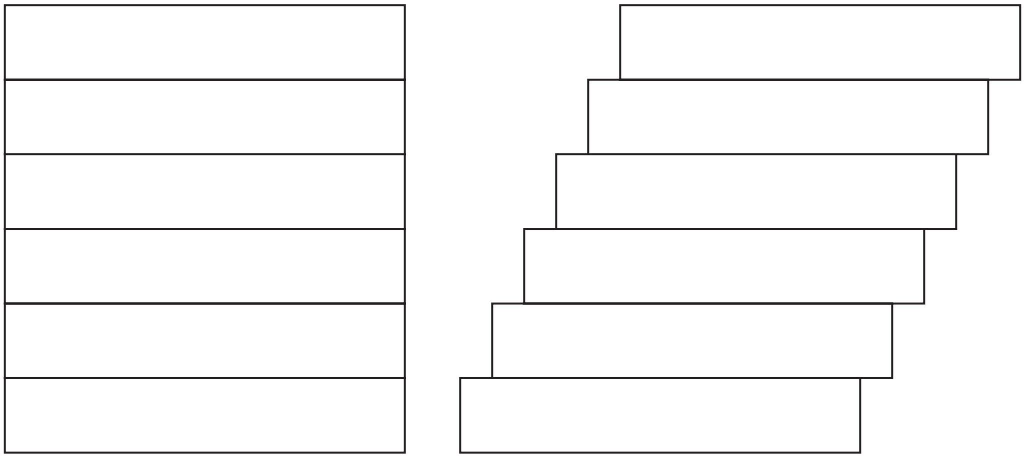
4.3 Applying Cavalieri's Principles

Cavalieri's principle for two-dimensional figures states that if the lengths of one-dimensional slices—just a line segment—of two figures are the same, then the figures have the same area.

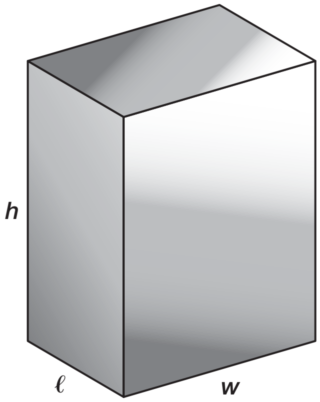
Cavalieri's principle for three-dimensional figures states that if, in two solids of equal altitude, the sections made by planes parallel to and at the same distance from their respective bases are always equal, then the volumes of the two solids are equal.

Examples

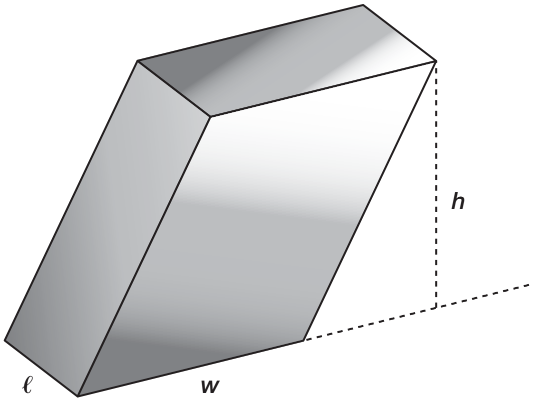
Both figures have the same area:



Both solids have the same volume:



Right rectangular prism



Oblique rectangular prism

4.4 Building the Cylinder Volume Formula

To build a volume formula for a cylinder, you can think of it as an infinite stack of discs, each with an area of πr^2. These discs are stacked to a height of h (the height of the cylinder).

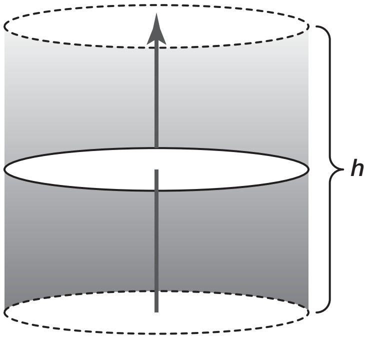
A cylinder can also be created by rotating a rectangle about one of its sides. To determine the volume formula for the cylinder, determine the average, or typical, point of the rectangle, which is located at $\left(\frac{1}{2}r, \frac{1}{2}h\right)$.

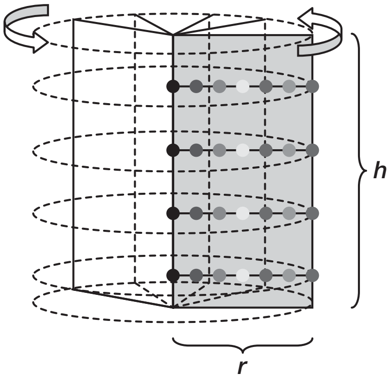
Then, use this point as the radius of a circle and calculate the circumference of that circle: $2\pi\left(\frac{1}{2}r\right) = \pi r$. This is the average distance that all the points of the rectangle are rotated to create the cylinder.

Finally, multiply this average distance by the area of the rectangle, rh, to determine the volume formula: $\pi r(rh) = \pi r^2h$.

Examples

The volume of any cylinder is $\pi r^2 \times h$, or πr^2h.





4.4 Building the Cone Volume Formula

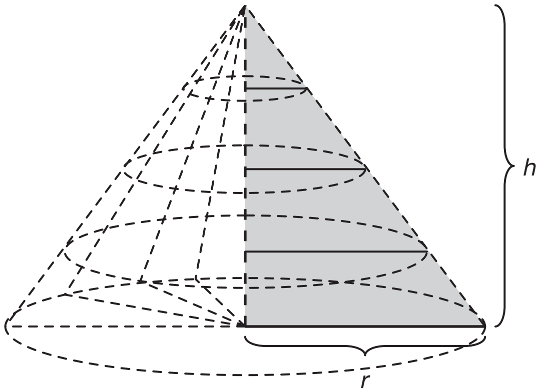
You can think of the volume of a cone as a stacking of an infinite number of similar discs at a height of h. You can also think of a cone as being created by rotating a right triangle.

To determine the volume formula for the cone, determine the average, or typical, point of the triangle, which is located at $\left(\frac{1}{3}r, \frac{1}{3}h\right)$. Then, use this point as the radius of a circle and calculate the circumference of that circle: $2\pi\left(\frac{1}{3}r\right) = \frac{2}{3}\pi r$. This is the average distance that all the points of the triangle are rotated to create the cone.

Finally, multiply this average distance by the area of the triangle, $\left(\frac{1}{2}rh\right)$, to determine the volume formula: $\frac{2}{3}\pi r \times \frac{1}{2}(rh) = \frac{1}{3}\pi r^2 h$.

Example

The volume of any cone is $\frac{1}{3}\pi r^2 h$.



4.4 Determining the Pyramid Volume Formula

A pyramid is to a prism what a cone is to a cylinder. A prism is created by stacking congruent polygons, which is similar to creating a cylinder by stacking congruent circles. A pyramid is created by stacking similar polygons that are not congruent, which is similar to creating a cone by stacking similar circles. The volume of any pyramid is $\frac{1}{3}$ the volume of a prism with an equal base area and height.

Example

Volume of prism: $V = Bh$

Volume of pyramid: $V = \frac{1}{3}Bh$

4.5 Calculating Volume of Spheres

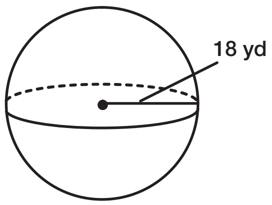
A sphere is the set of all points in three dimensions that are equidistant from a given point called the center.

The volume of a sphere is the amount of space contained inside the sphere. To calculate the volume of a sphere, use the formula $V = \frac{4}{3}\pi r^3$, where V is the volume of the sphere and r is the radius of the sphere.

The volume of a sphere is equal to twice the volume of a cylinder minus the volume of a cone with an equal base area and height.

Volume formulas can be used to solve problems.

Examples



$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi(18^3) \\ &\approx \frac{4}{3}(3.14)(5832) \\ &\approx 24{,}416.6 \end{aligned}$$

The volume of the sphere is approximately 24,416.6 cubic yards.

Volume of sphere $=$ 2 × Volume of cylinder $-$ Volume of cone

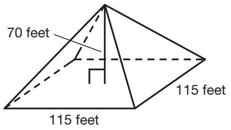
$$2 \times \pi r^2 h - \frac{1}{3}\pi r^2 h = 2 \times \frac{2}{3}\pi r^2 h = \frac{4}{3}\pi r^2 h$$

In a sphere, h and r are equal, so the volume formula for a sphere can be written as $\frac{4}{3}\pi r^3$.

4.6 Solving Problems Using Volume Formulas

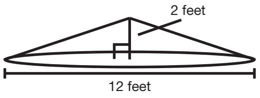
Formulas for the volumes of pyramids, cylinders, and cones can be used to solve problems.

Examples



$$V = \frac{1}{3}(115^2)(70) \approx 308{,}583$$

The volume of this pyramid is about 308,583 cubic feet.



$r = 6$, $h = 2$

$$V = \frac{1}{3}\pi(6^2)(2) = \frac{1}{3}(3.14)(36)(2) = 75.36$$

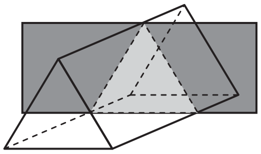
The volume of the cone is about 75.36 cubic feet.

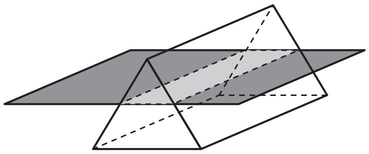
4.7 Determining Shapes of Cross Sections

A cross section of a solid is the two-dimensional figure formed by the intersection of a plane and a solid when a plane passes through the solid.

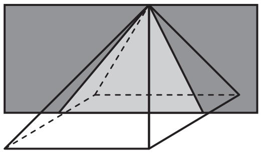
Examples

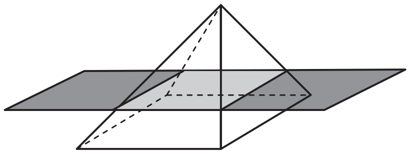
The cross sections of the triangular prism are triangles and rectangles.



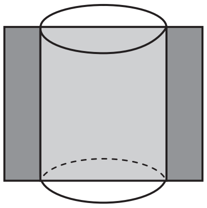


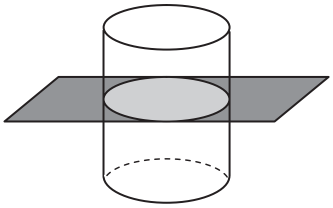
The cross sections of the square pyramid are triangles and squares.



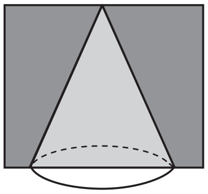


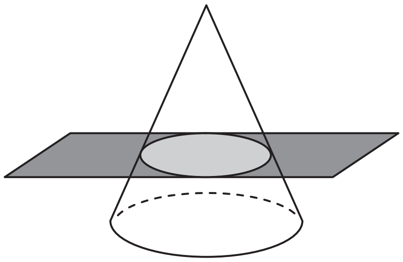
The cross sections of the cylinder are circles and rectangles.





The cross sections of the cone are circles and triangles.



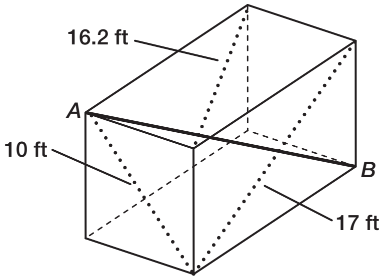


4.8 Determining Lengths of Spatial Diagonals

To determine the length of a spatial diagonal of a rectangular solid given the lengths of the diagonals of the faces of the solid, use the following formula, where d represents the length of a spatial diagonal, and d_1, d_2, and d_3 represent the lengths of the diagonals of the faces of the rectangular solid.

$$d^2 = \frac{1}{2}(d_1^2 + d_2^2 + d_3^2)$$

Examples



$$(AB)^2 = \frac{1}{2}(d_1^2 + d_2^2 + d_3^2)$$

$$(AB)^2 = \frac{1}{2}(10^2 + 17^2 + 16.2^2)$$

$$(AB)^2 = \frac{1}{2}(100 + 289 + 262.44)$$

$$(AB)^2 = \frac{1}{2}(651.44)$$

$$AB \approx 18$$

The length of the spatial diagonal AB is approximately 18 feet.

4